GROUP CLASSIFICATION OF ALGEBRAIC EQUATIONS BY ROOT PROPERTIES

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1. Let us consider the family of algebraic equations

$$P_n(z,\ a) \equiv z^n + a_1 z^{n-1} + \ldots + a_n = 0, \quad z \in K; \qquad (a_1 \ldots, a_n) \in D \tag{1.1}$$

for a fixed n ; K is the complex plane, D is the real, euclidean space. A number of stability and automatic control problems involve, as is well-known, the study of the properties of the roots of Equation (1.1) as dependent on its coefficients (considered as parameters).

In this paper we study a certain class Ω of such properties.

Let us assume that the conditions for their preservation may be expressed through the conditions of invariance of a finite number of differentiable relations between $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ relative to a certain Lie group G of transformations of the field K into itself. Thus, the preservation of the Hurwitz conditions is ensured by the condition for the invariance of Equation $x = 0$ under all transformations in some group G_1' (the transformations in G_1' leave the imaginary axis of the z-plane fixed); the condition that the reality of the roots be not violated is ensured the condition for the invariance of the equation $y = 0$ under all transformations in some (other) group G_2' (the transformations in G_2' leave the real axis fixed); the condition for the preservation of a (finite) number of roots of any equation will be fulfilled if the z-plane is subjected to transformations from any (locally) continuous group of transformations, etc.

An example of a property not belonging to class Ω is a property that the root of the equation is expressed rationally or in radicals in terms of the coefficient a .

The points of space D for which a given property $\omega \in \Omega$ is satisfied will be called ω-equivalent, and the whole set of such points, the ω-equivalent region.

The aim of this paper is to determine the boundaries of equivalence region. Certain related questions are also considered.

The paper is based on the following idea.

Together with a certain group G of transformations of field K into itself, let there exist a group G_a, isomorphic to it, of transformations of space D into itself (so that the transformations in G are independent of a, while the transformations in G_a are independent of z); moreover, G and G_a are such that the transformations in the extended group $G + G_a$ preserve Equation (1.1). Then, to every relation in K which is invariant relative to G there corresponds a certain relation (one, or several) in D which is invariant relative to G_a, and between the corresponding systems of intransitivity in K and D there is established a one-to-one correspondence.

This fact also holds for any pair of isomorphic subgroups $G' \leftrightarrow G_a'$, $G' \subset G$, $G_a' \subset G$. If group G is infinite and $\omega \in \Omega$, then we can find a subgroup $G' \subset G$, preserving this property and a subgroup $G_a' \subset G_a$ isomorphic to it.

Group G is said to be fundamental.

By what has been asserted above, ω is realized if and only if the ω-equivalence region coincides with the strictly defined systems of intransitivity of group G_a'. If group G_a' is transitive, then the determination of the desired boundaries reduces to the finding of particular invariant manifolds in G_a'. The latter can be done by standard algebraic ways. The aim of the paper is attained by indicating the base of the Lie algebra which allows us to solve in closed form certain problems of finding invarient manifolds in G_a'. By transitivity is always to be understood local transitivity at points of common location.

Within the framework of Lie algebras let us consider infinitesimal operators. We assume that it is possible to extend certain facts in the theory of finite Lie groups, used in this paper [1 and 2], to infinite groups. In particular, the fact that there exists a group corresponding to an infinite algebra. For simplicity, all the functions we shall encounter are assumed to be analytic (it suffices to consider them to be trice differentiable [1]). The dummy indices are summed everywhere, unless specified to the contrary. Quantities raised to powers are placed within paranthesis.

2. Let us consider the equation

$$f(z, a) = 0, \ z \in K, \ a = (a_1, \ldots, a_n) \in D \tag{2.1}$$

where it is known that the number of solutions it has equals the number of parameters a for any $a \in D$. Let $\lambda^1, \ldots, \lambda^n$ be the roots of Equation (2.1).

In accordance with generally applicable definitions, the parameters a are said to enter into the expressions for the λ^i in an essential wav if

$$\det \| f_j^i \| \neq 0, \qquad f_j^i = \frac{\partial f(\lambda^i, a)}{\partial a_j} \qquad (i, j = 1, \ldots, n) \tag{2.2}$$

for the point $a \in D$ of common location. Otherwise, there exists at least one system of functions

$$\zeta^1(a), \ldots, \zeta^n(a) \tag{2.3}$$

not all of which are identically zero, such that the identity

$$\zeta^j(a) \frac{\partial f(\lambda^i, a)}{\partial a_j} = 0 \qquad (i = 1, \ldots, n)$$

holds.

L e m m a 2.1 . If all the parameters of Equation (2.1) are essential, then there exist an infinite group G of transformations of K into itself and a group G_* isomorphic to it of transformations of D into itself, such that the extended group $G + G_*$ preserves Equation (2.1); group G_* is transitive, while group G is multiply transitive any number of times.

P r o o f . Let us denote by

$$Z_i = \xi_i(z)\frac{\partial}{\partial z}, \qquad A_i = \zeta_i^j(a)\frac{\partial}{\partial a_j}, \qquad X_i = Z_i + A_i \qquad (i = 1, \ldots, n)$$

respectively, the infinitesimal operators in groups G, G_* and $G+G_*$. In order to obtain the infinite Lie algebra $\mathfrak{G}$ of group G, obviously, it is sufficient to choose as $\xi_i(z)$ any base in a space of sufficiently wide class of functions. Let such a base be chosen in the class of analytic functions. Because of the linear independence of the functions $\xi_i(z)$, for every finite k the rank of the matrix

$$\begin{Vmatrix} \xi_1(u^1) \ldots \xi_1(u^k) \\ \xi_2(u^1) \ldots \xi_2(u^k) \\ \ldots\ldots\ldots\ldots \end{Vmatrix}$$

equals k , whence follows the k-fold transitivity of group G .

The components $\zeta_*^j(a)$ are defined by the equalities

$$\zeta_m^j(a)\frac{\partial f(z, a)}{\partial a_j} + \xi_m(z)\frac{\partial f(z, a)}{\partial z}\bigg|_{z=\lambda^i(a)} = 0 \qquad \begin{pmatrix} i, j = 1, \ldots, n \\ m = 0, 1, \ldots \end{pmatrix} \tag{2.4}$$

where $\lambda^i = \lambda^i(a)$ are the solutions of (2.1). Because of (2.2) there also exist unique functions $f^{*\beta}_j$ satisfying the equations

$$f_i^j f^{*\beta}_j = \delta_i^\beta \qquad (\delta_i^\beta \text{ – Kronecker symbol})$$

Solving (2.4) we obtain uniquely

$$\zeta_m^j(a) = -\sum_{i=1}^{n} f^{*j}_i\left[\xi_m(z)\frac{\partial f(z, a)}{\partial z}\right]_{z=\lambda^i(a)} \tag{2.5}$$

Because $f(z,a) = 0$, it follows from (2.4) that

$$X_m f(z, a) = 0 \tag{2.6}$$

If $X_\alpha f(z, a) = 0$ and $X_\beta f(z, a) = 0$ by virtue of $f(z, a)=0$, then

$$(e_1X_\alpha + e_2X_\beta) f(z, a) = 0$$

and it is also easy to show that

$$(X_\alpha, X_\beta) f(z, a) = 0 \tag{2.7}$$

because of the same relations.

Indeed, if (2.6) is valid, then we can find a function $\mu(z,a)$ and an integer s such that

$$Xf(z, a) \equiv \mu(z, a)\,[f(z, a)]^s$$

Then

$$(X_\alpha, X_\beta) f \equiv (s_\beta - s_\alpha)\,\mu_\alpha\mu_\beta f^{s_\alpha+s_\beta-1} + f^{s_\beta}X_\alpha\mu_\beta - f^{s_\alpha}X_\beta\mu_\alpha$$

which, for sufficiently general constraints on $\mu(z,a)$, also proves (2.7).

Consequently, the operators X_s do indeed generate the Lie algebra $\mathfrak{G} + \mathfrak{G}_*$. From the uniqueness of solution (2.5) we can conclude that the algebras $\mathfrak{G}$ and $\mathfrak{G}_*$ are isomorphic. Relations (2.6) show that the transformations in the group $G + G_*$ preserve Equation (2.1). From (2.5) we get

$$\det \| \zeta^m (a) \| = - \det \| f^{*j}_{i} \| \cdot \det \| \xi_m (\lambda^i) \| \cdot \prod_{\gamma=1}^{n} \left. \frac{\partial f(z, a)}{\partial z} \right|_{z=\lambda^\gamma (a)} \neq 0 \qquad (m = m_1, \ldots, m_n) \tag{2.8}$$

for any point $a \in D$ of common location, since (2.2) holds, group G is multiply transitive and at the point a of common location Equation (2.1) cannot have multiple roots (and hence $\partial f/\partial z \neq 0$ when $z = \lambda^\gamma$). From (2.8) the transitivity of group G_* follows. Lemma 2.1 is proved.

Lemma 2.1 is easily generalized to the case of a system of equations.

Let us note without proof the validity of a somewhat more general assertion: if Equation (2.1) contains $\rho > n$ parameters of which n are essential, then there exists an infinite group G of transformations of K into itself, and its homomorphic inverse image, the group G_* of transformations of D into itself; these groups are such that the extended group $G + G_*$ preserves Equation (2.1), and, moreover, group G is multiply transitive any number of times, while group G_* is intransitive and admits of precisely $\rho - n$ absolute invariants.

To the kernel of homomorphism H there corresponds the ideal $\mathfrak{H}$ in algebra $\mathfrak{G}_*$ generated by $\rho - n$ operators with components (2.3), and, according to a well-known Theorem in [1],(see p.22) the group G is isomorphic with the factor group G_*/H.

Since Lemma 1 is valid for any choice of infinite algebra $\mathfrak{G}$ then for a given choice of $\mathfrak{G}$ it remains valid also for every infinite subalgebra $G' \subset G$ and, consequently, there corresponds to it the finite subgroup $G' \subset G$ if such exist.

Let us sharpen the definition given in Section 1 of the class Ω of the properties being investigated of the roots of Equation (2.1). Let us consider all possible systems of curves in the z-plane

$$\Phi_i (x, y) = 0 \qquad (i = 1, \ldots, l) \tag{2.9}$$

Let one of these systems be such that there exists such an infinite collection of mutually conjugate harmonic functions $\xi(x,y)$, $\eta(x,y)$ that by virtue of $\Phi_i = 0$

$$Y\Phi_i \equiv \xi(x, y) \frac{\partial \Phi_i}{\partial x} + \eta(x, y) \frac{\partial \Phi_i}{\partial y} = 0 \tag{2.10}$$

The class Ω consists of the properties of the roots of Equation (2.1) which preserve their distribution between regions of the z-plane bounded by any system of curves (2.9), including distribution between the curves themselves and their arcs *.

* Translator's Note: There would be many possible translations depending on which inflection was incorrect. It is hoped the intended interpretation is given here.

In Section 3 it will be proved that class Ω is not empty.

From this definition it automatically follows that if $\omega \in \Omega$, then there exists an infinite subgroup $G' \subset G$ preserving property ω.

In a certain sence the properties in class Ω are "locational properties". It is easy to imagine, however, properties of a more complex nature which are of interest in application. These are properties which are expressed in the form of certain relations between the real and imaginary parts of the roots of the equation. For example, such a property is that of the specified alternations of real and pairs of complex-conjugate roots. In this case the transformations in group G themselves should be dependent on the location of the roots in the z-plane, i.e. on a.

In the considered example let $x_1 + iy_1$, $x_1 - iy_1$ and x_2 be some roots of Equation (2.1). Let us introduce the notations $\psi_1 \equiv x_1 - x_2$, $\psi_2 \equiv \operatorname{Re} f(z, a)$, $\psi_3 \equiv \operatorname{Im} f(z, a)$.

Transformations of the z-plane will not disturb the nature of the root alternations if we require the fulfillment of the condition $Y\psi_1 = 0$ by virtue of $\psi_1 = \psi_2 = \psi_3 = 0$.

This requirement isolates from G an infinite subgroup which preserves root alternations. It is not difficult to construct this subgroup.

L e m m a 2.2 . If a certain property $\omega \in \Omega$ of the roots of Equation (2.1) is satisfied at a given fixed point a°, then it is also satisfied in the whole intransitivity region $M \ni a^\circ$ of the subgroup $G_a' \subset G_a$ preserving ω.

P r o o f . From (2.2) there follows a one-to-one correspondence between λ^i and a. Because of (2.4) this correspondence is preserved also for $\lambda^{i\prime} \leftarrow \lambda^i$, $a' \leftarrow a$, obtained from λ^i and a by means of all possible transformations in the group $G + G_*$. Therefore, if it should turn out that there exists a point $a^1 \in M$ for which the corresponding λ^i do not possess property ω, then this would show that the transformations in group G' do not preserve ω, which contradicts the hypothesis.

From Lemmas 2.1 and 2.2 and from the method of obtaining the equations of singular invariant manifolds of transitive groups, there follows a theorem.

T h e o r e m 2.1 . The ω-equivalence regions ($\omega \in \Omega$) coincide with the intransitivity systems in G_*', preserving ω, which are n-dimensional and which are decomposable by hypersurfaces on which vanish all the n-th order minors of the vector matrix $\|\zeta_*^{\,j}(a)\|$ of group G_*'.

3. Let us consider the question of choosing G and of the computation of the algebra G_* based on the choice.

Equation (1.1) satisfies the condition of Lemma 2.1. Let us find the algebra $G + G_*$ for it. Let us write G in the form

$$Z_m \equiv z^m \frac{\partial}{\partial z} \qquad (m = 0, 1, 2, \ldots) \tag{3.1}$$

The operators (3.1) indeed generate the algebra since

$$(Z_m, Z_l) = (l - m) Z_{l+m-1}$$

The components $\zeta_*^{\,j}(a)$ of the operators in algebra G_* are found in the following way. In order that Equation (1.1) admit of group $G + G_*$ it is sufficient that for every integer m there is found a polynomial of degree $m-1$

$$p_{m-1}(z, a) = \sum_{\nu=0}^{m-1} \alpha_{m-\nu} z^\nu$$

such that the condition

$$X_m P_n \equiv z^m \frac{\partial P_n}{\partial z} + \zeta_m^j(a) \frac{\partial P_n}{\partial a_j} \equiv p_{m-1} P_n \tag{3.2}$$

is identically fulfilled with respect to z and a.

By substituting the left-hand side of Equation (1.1) into (3.2), we get

$$\sum_{h=m}^{n+m-1} (h-m+1) a_{n+m-h-1} z^h + \sum_{h=0}^{n-1} \zeta_m^{n-h}(a) z^h = \sum_{h=0}^{n+m-1} \Big(\sum_{k+\nu=h} a_{n-k} \alpha_{m-\nu-1} \Big) z^h$$

Hence by equating to zero the alternate coefficients for various powers of z, we find

when $m \leqslant n-1$

$$\zeta_m^{n-h}(a) = \sum_{k+\nu=h} a_{n-k} \alpha_{m-\nu-1} \qquad (0 \leqslant h \leqslant m-1)$$

$$\zeta_m^{n-h}(a) = \sum_{k+\nu=h} a_{n-k} \alpha_{m-\nu-1} - (h-m+1) a_{n+m-h-1} \qquad (m \leqslant h \leqslant n-1)$$

$$0 = \sum_{k+\nu=h} a_{n-k} \alpha_{m-\nu-1} - (h-m+1) a_{n+m-h-1} \qquad (n \leqslant h \leqslant n+m-1)$$

when $m \geqslant n$

$$\zeta_m^{n-h}(a) = \sum_{k+\nu=h} a_{n-k} \alpha_{m-\nu-1} \qquad (0 \leqslant h \leqslant n-1)$$

$$0 = \sum_{k+\nu=h} a_{n-k} \alpha_{m-\nu-1} \qquad (n \leqslant h \leqslant m-1)$$

$$0 = \sum_{k+\nu=h} a_{n-k} \alpha_{m-\nu-1} - (h-m+1) a_{n+m-h-1} \qquad (m \leqslant h \leqslant n+m-1)$$

It is not difficult to verify that for arbitrary m the corresponding systems give unique solutions for $\zeta_*^j(a)$ in the form of polynomials in a.

The very same result is obtained by considering the Viet formula

$$\varphi_1 \equiv a_1 + (\lambda^1 + \ldots + \lambda^n) = 0, \ldots, \varphi_n \equiv a_n + (-1)^{n+1} \lambda^1 \ldots \lambda^n$$

and by taking into account that because $\varphi_1 = \varphi_2 = \ldots = \varphi_n = 0$

$$X_i \varphi_j = 0$$

Thus, because $\varphi_1 = \ldots = \varphi_n = 0$ we get

$$\zeta_m{}^1(a) + (\lambda^1)^m + \ldots + (\lambda^n)^m = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\zeta_m{}^n(a) + (-1)^{n+1}[(\lambda^1)^m \ldots \lambda^n + \ldots + \lambda^1 \ldots (\lambda^n)^m] = 0$$

The collection of terms on the left-hand sides, depending on λ^1, by virtue of the theorem of symmetric polynomials, rationally is expressed in terms of basic symmetric polynomials, i.e. in terms of $a_1, \ldots, a_n$.

Actually, it is not necessary to compute the functions $\zeta_m{}^j(a)$ by the cited formulas for large values of m; besides the operators A_0, A_1, A_2 corresponding to the bilinear group of transformations of z, it suffices only to find A_3. After this, the remaining operators A may be computed by using the commutations of A_m with A_3 $(m = 2,3,\ldots)$. The components of A_3 may be determined from Formulas

$$\zeta_3^j(a) = \sum_{k+\nu+j=n} a_{n-k}\alpha_{2-\nu} - (n-j-2)\,a_{2+j} \qquad (1 \leqslant j \leqslant n-3)$$

$$\zeta_3^j(a) = \sum_{k+\nu+j=n} a_{n-k}\alpha_{2-\nu} \qquad (n-2 \leqslant j \leqslant n)$$

$$(\alpha_0 = a_1{}^2 - 2a_2, \quad \alpha_1 = -a_1, \quad \alpha_2 = n)$$

In many cases it is convenient to pass from the variables $a_1, \ldots, a_n$ to the equivalent system of variables

$$s_k = \sum_{i=1}^{n} (\lambda^i)^k \qquad (k = 1, \ldots, n)$$

by use of Newton's recurrence formulas. Then, instead of algebra G_* we obtain the algebra G_* isomorphic to it

$$\zeta_m{}^k(s) = \sum_{i=1}^{n} k\,(\lambda^i)^{k-1}(\lambda^i)^m = \sum_{i=1}^{n} k\,(\lambda^i)^{k+m-1} = ks_{k+m-1} \tag{3.3}$$

The elements of algebra $G + G_*$ take the form

$$X_m = z^m \frac{\partial}{\partial z} + \sum_{k=1}^{n} ks_{k+m-1}\frac{\partial}{\partial s_k} \tag{3.4}$$

By using the base of G we can construct subalgebras which are useful when considering certain transcendental equations satisfying the condition of Lemma 2.1. The base

$$\cos mz\frac{\partial}{\partial z}, \qquad \sin mz\frac{\partial}{\partial z} \qquad (m = 0, 1, \ldots) \tag{3.5}$$

can be formally obtained from (3.1) by the summations

$$\cos mz = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!}\,\xi_{2k}(mz), \qquad \sin mz = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k-1)!}\,\xi_{2k-1}(mz)$$

or by passing to a new variable in the old base. In particular, in the same way we can obtain also the base

$$e^{mz}\frac{\partial}{\partial z} \tag{3.6}$$

The application of bases (3.5) and (3.6) to Equation (1.1) leads to transcendental expressions for the components $\zeta_s^j(a)$ of algebra G_s

The base (3.6) can be used when studying Equation

$$\sum_{\nu=0}^{n} a_{n-\nu} e^{\nu z} = 0, \qquad a_0 = 1$$

for obtaining the $\zeta_s^j(a)$ in the form of polynomials in a . However, the written equation does not essentially differ from Equation (1.1) and can be obtained from it by passing to the other variable $z \to e^z$, in the same way that the algebra of the exponential equation is obtained from the algebra of Equation (1.1).

Of great interest is the direct determination of algebra G_s from (3.5) for the trigonometrical equation

$$f(z, a) \equiv a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu z + b_\nu \sin \nu z) = 0 \qquad (a_n^2 + b_n^2 = 1) \tag{3.7}$$

By requiring the fulfillment of (3.2), where it should be assumed that

$$P_n = f, \qquad \xi_m(z) = \sin mz, \qquad P_{m-1} = \alpha_0 + \sum_{\mu=1}^{m} (\alpha_\mu \cos \mu z + \beta_\mu \sin \mu z)$$

we get

$$\begin{aligned}
&\tfrac{1}{2}\Big[\sum_{\omega=m+1}^{m+n} (\omega - m)\, a_{\omega-m} \cos \omega z - \sum_{\omega=m-1}^{m-n} (m-\omega)\, a_{m-\omega} \cos \omega z + \\
&+ \sum_{\omega=m+1}^{m+n} (\omega - m)\, b_{\omega-m} \sin \omega z - \sum_{\omega=m-1}^{m-n} (m-\omega)\, b_{m-\omega} \sin \omega z\Big] + \\
&+ \zeta_0(a, b) + \sum_{\omega=1}^{n} [\zeta_{am}^{\omega}(a, b) \cos \omega z + \zeta_{bm}^{\omega}(a, b) \sin \omega z] = \\
&= \alpha_0 a_0 + a_0 \sum_{\omega=1}^{m} (\alpha_\omega \cos \omega z + \beta_\omega \sin \omega z) + \alpha_0 \sum_{\omega=1}^{n} (a_\omega \cos \omega z + b_\omega \sin \omega z) + \\
&+ \tfrac{1}{2} \sum_{\omega=2}^{m+n} \sum_{\nu+\mu=n} [(a_\nu \alpha_\mu - b_\nu \beta_\mu) \cos \omega z + (b_\nu \alpha_\mu + a_\nu \beta_\mu) \sin \omega z] + \\
&+ \tfrac{1}{2} \sum_{\omega=1-m}^{n-1} \sum_{\nu-\mu=\omega} [(a_\nu \alpha_\mu + b_\nu \beta_\mu) \cos \omega z + (b_\nu \beta_\mu - a_\nu \alpha_\mu) \sin \omega z]
\end{aligned}$$

Hence, by equating to zero alternate coefficients of $\cos \omega z$ and $\sin \omega z$, we obtain a unique solution in the form of polynomials in a and b for the functions ζ_{as}^j and ζ_{bs}^j generating the algebra

$$\sin mz \frac{\partial}{\partial z} + \zeta_{am}^j(a, b) \frac{\partial}{\partial a_j} + \zeta_{bm}^j(a, b) \frac{\partial}{\partial b_j}$$

The computations for $\xi_s(z) = \cos mz$ are done analogously. The number of roots of Equations (3.7) is (denumerably) infinite, but we can satisfy the condition of Lemma 2.1 if we mutually identify all the roots of this equation having equal imaginary parts and having real parts differing by multiples of 2π .

For the system of algebraic equations

$$P_n^i(z^1, \ldots, z^k, a) = 0 \qquad (i = 1, \ldots, k)$$

the generators $\xi_m^1(z^1, \ldots, z^k); \ldots ; \xi_m^k(z^1, \ldots, z^k)$ of algebra $\mathfrak{G}$ can be taken as monomials of degree m in the variables $z^1, \ldots, z^k$. Then, the generators $\zeta_s^j(a)$ of algebra $\mathfrak{G}_s$ will be polynomials in $a_1^i, \ldots, a_n^i$.

As can be seen from what has been presented, the form of algebra $\mathfrak{G}_s$ essentially depends on the choice of algebra $\mathfrak{G}$.

It is natural to take advantage of the arbitrariness in the choice of $\mathfrak{G}$ so that the expressions for the components $\zeta_s^j(a)$ of algebra $\mathfrak{G}_s$ obtained, would have as simple as possible form (although the reduction of these components to polynomials is not always possible).

The second essential requirement on $\mathfrak{G}$ is the possibility of deriving from $\mathfrak{G}$ an infinite subalgebra preserving the properties under study of the roots of Equation (2.1) (in particular, of (1.1)).

4. Let us make use of the algebra

$$\xi_m(z)\frac{\partial}{\partial z} \equiv z^m \frac{\partial}{\partial z}$$

considered in Section 3 for the study of certain properties of the roots of Equation (1.1).

Group (3.1) realizes the following representation in the plane $z = x + iy$. Let τ be the canonic parameter of any one-parameter subgroup of G . Then, from the Lie equation $\partial z'/\partial\tau = z'^m$ follows

$$\frac{\partial x'}{\partial \tau} + i\frac{\partial y'}{\partial \tau} = \operatorname{Re} z'^m + i \operatorname{Im} z'^m$$

By equating the real and imaginary parts in this equation we get the algebra of the desired representation

$$\xi_m(x, y)\frac{\partial}{\partial x} + \eta_m(x, y)\frac{\partial}{\partial y}, \qquad \xi_m(x, y) = \operatorname{Re} z^m, \qquad \eta_m(x, y) = \operatorname{Im} z^m$$

In polar coordinates ρ, φ $(z = \rho e^{i\varphi})$,

$$\eta_{\rho m}(\rho, \varphi)\frac{\partial}{\partial \rho} + \eta_{\varphi m}(\rho, \varphi)\frac{\partial}{\partial \varphi} \qquad (m = 0, 1, \ldots) \tag{4.1}$$

$$\eta_{\rho m}(\rho, \varphi) = \rho^m \cos(m-1)\varphi, \qquad \eta_{\varphi m}(\rho, \varphi) = \rho^{m-1}\sin(m-1)\varphi$$

Let us use the explicit expressions for $\eta_{\rho m}$ and $\eta_{\varphi m}$ to separate out that infinite subgroup which preserves all of the properties of the roots of Equation (1.1) considered below.

a) R e a l i t y o f t h e r o o t s . The number of real roots of Equation (1.1) remains fixed under all transformations in a certain group $G' \subset G$ if they preserve the equation $\varphi = 0$. Consequently, the Lie algebra of group G' is generated by those operators (4.1) for which the condition

$$\eta_{\rho m}(\rho, \varphi)\frac{\partial\varphi}{\partial\rho} + \eta_{\varphi m}\frac{\partial\varphi}{\partial\varphi} = 0$$

is satisfied by virtue of $\varphi = 0$.

But this is satisfied identically for all integers m so that G' coincides with the fundamental group $G' = G$.

According to Theorem 2.1, to obtain the equation of the hypersurface bounding the region in which the number of real roots is constant, it is necessary to find the greatest divisor of all the n-th order minors of the matrix $\|\zeta_s{}^j(a)\|$.

According to (3.4) this matrix has the form

$$\left\| \begin{array}{cccc} s_0 & 2s_1 & \cdots & ns_{n-1} \\ s_1 & 2s_2 & \cdots & ns_n \\ \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & 2s_n & \cdots & ns_{2n-1} \\ \cdots & \cdots & \cdots & \cdots \end{array} \right\|$$

Let us restrict our attention to the upper minor. At any rate, it contains as a factor the desired greatest divisor

$$\begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \end{vmatrix} = 0 \qquad (4.2)$$

The Hankel determinant on the left-hand side of (4.2) will be an irreducible polynomial and will coincide with the resultant of the system of equations

$$P_n(z, a) = 0, \qquad \frac{\partial P_n(z, a)}{\partial z} = 0$$

In space D the hypersurface (4.2) bounds a certain multicavity region. Each cavity is an equivalence region relative to the number of real roots. To separate out those cavities which correspond to given numbers of real roots, it is necessary to have additional conditions. Such a condition, for the case when all the roots are real, is the positiveness of all the diagonal minors of determinant (4.2), as is well-known. To obtain these conditions (if a priori knowledge of them is not assumed) it is necessary to study the geometry of manifold (4.2) and, for any point representative of each cavity, to establish to what number of real roots it corresponds.

b) Stability of the roots. The Hurwitz conditions will not be violated if the z-plane is subjected to all possible transformations which leave the imaginary axis fixed. The corresponding Lie algebra is generated by those of operators (4.1) for which condition

$$\eta_{\rho m}(\rho,\varphi)\frac{\partial\varphi}{\partial\rho}+\eta_{\varphi m}(\rho,\varphi)\frac{\partial\varphi}{\partial\varphi}=0$$

is satisfied by virtue of the equality $\varphi - \frac{1}{2}\pi = 0$.

Therefore, we should have

$$\eta_{\varphi m}(\rho, {}^1/_2\pi) = \rho^{m-1}\sin\left[{}^1/_2(m-1)\pi\right] = 0 \quad \text{or} \qquad m = 2k+1 \ (k=0,1,\ldots)$$

Thus, the desired subalgebra G' is generated by the system of operators (3.1) with an odd index m. The matrix of subalgebra G_s' has the form

$$\left\| \begin{matrix} s_1 & 2s_2 & \ldots & ns_n \\ s_3 & 2s_4 & \ldots & ns_{n+2} \\ \ldots & \ldots & \ldots & \ldots \\ s_{2n-1} & 2s_{2n} & \ldots & ns_{3n-1} \\ \ldots & \ldots & \ldots & \ldots \end{matrix} \right\|$$

It can be verified that the upper minor of the matrix is decomposable into the factors

$$\begin{vmatrix} s_1 & s_2 & \ldots & s_n \\ s_3 & s_4 & \ldots & s_{n+2} \\ \ldots & \ldots & \ldots & \ldots \\ s_{2n-1} & s_{2n} & \ldots & s_{3n-1} \end{vmatrix} = \begin{vmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ \ldots & \ldots & \ldots & \ldots \\ s_{n-1} & s_n & \ldots & s_{2n-1} \end{vmatrix} \cdot \begin{vmatrix} a_1 & 1 & 0 & \ldots & 0 \\ a_3 & a_2 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{vmatrix}$$

The second of these factors will be the Hurwitz determinant of the highest order and in turn it is decomposable into two irreducible factors. In certain cases their structure allows us to find the Hurwitz conditions themselves in a simple manner.

As an example let us consider the case of $n = 4$. By a test carried out for some fixed values of the roots located in the left-hand halfplane, we establish

$$a_4 \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} > 0$$

Consequently, both factors on the left-hand side of this inequality cannot change sign. A test of the signs of these factors, carried out for the same values of the roots, shows that both factors are positive. But

$$0 < \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} = a_3 \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} - a_4 a_1^2 \quad \text{or} \qquad a_3 \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > a_4 a_1^2 > 0$$

A repetition of the test shows that we should have

$$\begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > 0, \quad a_3 > 0, \quad a_1 a_2 > a_3 > 0, \quad a_1 > 0, \quad a_2 > 0$$

Such a method is applicable for any n. However, the completeness and the independence of the system of inequalities thus obtained should be checked each time. The last remark also relates to the case of the study of other ω by this method; in particular, it relates to the obtaining of various modifications and generalizations of the Hurwitz criterion.

c) Location of the roots within the angle $-\frac{1}{2}\mu\pi \leqslant \varphi \leqslant \frac{1}{2}\mu\pi$. Here μ is a rational fraction, $0 < \mu < 1$. Such a location of the roots is not violated by transformations which leave the pair of straight lines $\varphi = \pm\frac{1}{2}\mu\pi$ fixed.

It is easy verified that the corresponding Lie subalgebra consists of the operators (4.1) for which $m = 2k/\mu + 1$ (k/μ is an integer).

The cases $\mu = 0$ and $\mu = 1$ have already been considered in Subsections (a) and (b).

d) Location of the roots in the segment [0,1] of the real axis. This property of the roots is not violated by transformations which leave the real axis and the points $x = 0$ and $x = 1$, fixed. The subalgebra G' is generated by the operators

$$z^m (z-1)\frac{\partial}{\partial z} \qquad (m = 1, 2, \ldots)$$

The matix of algebra G_s' has the form

$$\begin{Vmatrix} 2s_2 - s_1 & 3s_3 - 2s_2 & \ldots & (n+1)s_{n+1} - ns_n \\ 3s_3 - 2s_2 & 4s_4 - 3s_3 & \ldots & (n+2)s_{n+2} - (n+1)s_{n+1} \\ 4s_4 - 3s_3 & 5s_5 - 4s_4 & \ldots & (n+3)s_{n+3} - (n+2)s_{n+2} \\ \ldots & \ldots & \ldots & \ldots \end{Vmatrix}$$

e) Alternation of the real roots of the pair of Equations

$$P_n(z, a) = 0, \qquad P_n(z, b) = 0$$

It is obvious that the nature of the alternation of the real roots of these equations is in no way disturbed by transformations in the fundamental group G . When the equations are of equal degree, the group of transformations of parameters a, b will be the doubled group $G_a + G_b$.

For example, for a pair of quadratic equations the matrix of this group has the form

$$\begin{Vmatrix} 2 & a_1 & 2 & b_1 \\ a_1 & 2a_2 & b_1 & 2b_2 \\ 2a_2 - a_1^2 & -a_1a_2 & 2b_2 - b_1^2 & -b_1b_2 \\ a_1^3 - 3a_1a_2 & a_1^2a_2 - 2a_2^2 & b_1^3 - 3b_1b_2 & b_1^2b_2 - 2b_2^2 \\ \ldots & \ldots & \ldots & \ldots \end{Vmatrix}$$

f) Location of the roots inside a horizontal or vertical strip in the z-plane. In the xy-plane the algebra (3.5) realizes a group with the operators

$$\cos mx \cosh my \frac{\partial}{\partial x} - \sin mx \sinh my \frac{\partial}{\partial y}, \qquad \sin mx \cosh my \frac{\partial}{\partial x} + \cos mx \sinh my \frac{\partial}{\partial y}$$

and algebra (3.6), a group with the operators

$$e^{mx}\cos my\frac{\partial}{\partial x}+e^{mx}\sin my\frac{\partial}{\partial y}$$

Hence it is seen that the transformations in the groups corresponding to these algebras leave fixed, respectively, the vertical strip $-\pi\leqslant x\leqslant\pi$ and the horizontal strip $-\pi\leqslant y\leqslant\pi$. The components $\zeta_*^j(a)$, in this case, are obtained as entire transcendental functions; expansions in power series in the parameter a may be written out for them.

g) Sign-definiteness of quadratic forms. Let us consider the quadratic form

$$\Phi=a_{ij}x^i x^j,\qquad a_{ij}=a_{ji}\qquad (i,j=1,\ldots,n)$$

It is obvious that transformations in the linear group

$$x^i=\alpha_{i_1}{}^i x^{i_1},\qquad \det\|\alpha_{i_1}{}^i\|\neq 0\qquad (i_1=1,\ldots,n)$$

cannot change the parity of the form Φ.

The finite algebra G_* obtained in this case has a matrix with the elements

$$(\partial a'_{i_1i_2}/\partial\alpha_i{}^j)_{\alpha_i{}^j=\delta_i{}^j}=a_{ji_1}\delta_{i_2}{}^i+a_{ji_2}\delta_{i_1}{}^i$$

The minors of this matrix (its rank equals $\frac{1}{2}n(n+1)$) have the determinant $\det\|a_{ij}\|$ as the greatest common divisor.

The equation $\det\|a_{ij}\|=0$ defines an irreducible manifold which is a bounded equivalence region relative to the parity of form Φ. The Sylvester conditions, in particular, separate out in D the equivalence regions relative to the property of sign-definiteness of form Φ.

Note. The results of Sections 2 to 4 can be extended to the case of complex parameters. If $a=\rho e^{i\varphi}$, then the transformations of parameters a_j are effected by a group G_* with the operators

$$(\operatorname{Re}\zeta_m^j(a)\cos\varphi+\operatorname{Im}\zeta_m^j(a)\sin\varphi)\frac{\partial}{\partial\rho}+\frac{1}{\rho}(\operatorname{Im}\zeta_m^j(a)\cos\varphi-\operatorname{Re}\zeta_m^j(a)\sin\varphi)\frac{\partial}{\partial\varphi}$$

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